Estimation of Modal Mass Using $H^\infty$ Optimal Model Reduction

Jae-Seung Hwang$^{1)}$, Hong-Jin Kim$^{2)}$

$^{1)}$Assistant Professor, Faculty of Architecture, Chonnam National University, Gwangju, Korea
Tel: 82-62-530-1641, Fax: 82-62-530-1639, e-mail: jshwang@jnu.ac.kr
$^{2)}$Researcher, Research Institute of Industrial Science & Technology (RIST), Kyungkido, Korea

ABSTRACT
Modal mass of structure is essential to analyze structural response under disturbance and to design the mass type control devices used to enhance the serviceability of structure. Modal mass of real structure differs from that of analytic mathematical model due to the error induced from analytical assumption and construction. In this study, a estimation method is proposed to calculate the modal mass of real structure based on $H^\infty$ optimal model reduction. The modal mass is obtained from the relationship between observability, controllability matrix of realized from the system identification and typical 2-degree state space model. The proposed method is verified thorough the three examples.

Keywords: Modal Mass, System Identification, $H^\infty$ Optimal Model Reduction, Mass Type Control Device, Controllability Matrix, Observability Matrix

1. Introduction

Modal analysis method that estimates the structural responses based on superposition of single-degree-of-freedom (SDOF) model is a traditional and powerful dynamic analysis method for linear multi-degree-of-freedom (MDOF) systems. Specially, it is a very effective method for building and civil structures whose behaviors are governed by a few lowest modal responses. For modal analysis, the eigenvectors, which form the vector space basis of structural response and are linearly independent, need to be identified. Since eigenvectors obtained from eigenvalue analysis have relative values in the modal space, they are normalized to certain values. The eigenvector normalization is achieved by setting an arbitrary element of each vector to ‘1’, setting the largest element of each vector to ‘1’, or setting the modal mass of each mode to ‘1’.

The modal mass (or generalized mass) of each mode has relative and varying value depending on the eigenvector normalization method. For modal mass to have physical significance, it is combined with other variables that have physical meaning and characterized by those variables. Consequently, the magnitude of modal mass depends on the normalization method and physical characteristics of other variables.

The mass of secondary mass type control device used for vibration suppression such as tuned mass damper (TMD), active mass damper (AMD), and hybrid mass damper (HMD) is determined depending primarily on the modal mass. Therefore, the modal mass should be identified accurately such that it is reflected in design of control device correctly according to device type and installation location. Further, the accurate modal mass identification is critical because the modal mass is required for the tuning of design parameters of control device, which is required for evaluation of control performance and for guaranteeing the optimal control effect after the device is installed.

The modal mass is defined for separation of modal space that has linearly independent basis, and its definition and analytical calculation methods are introduced in numerous literatures dealing with structural dynamics$^{1), 2), 3)}$.

System identification methods that estimate the dynamic characteristics of structure from shaking table test and recorded response time histories have been widely and actively studied recently. The analysis and experimental method for modal characteristics identification$^{4)}$, identification theory in time and frequency domains$^{5), 6)}$, identification of nonlinear system and real-time identification are developed$^{7), 8)}$, and a computer program is introduced for easy system identification$^{9)}$. Recently, many researches on the system identification of high-rise buildings under wind load have been performed$^{10)}$. However, they concentrate on the estimation of state space equation, vibration frequency, damping ratio, and modal shape, and research on the identification of modal mass has rarely performed.

In this paper, a modal mass identification method is proposed for accurate structural analysis and precise performance estimation of control devices. The existing modal mass identification methods are briefly reviewed and a new method is presented for
identification of real structures based on vibration test and system identification method.

The proposed method extracts the modal mass from dynamic properties estimated using the input (excitation) and the output (responses). The modal frequency and damping ratio are obtained first by conventional system identification technique, and the state space equation for the equivalent single-degree-of-freedom (SDOF) system is established. The modal mass is then obtained by comparing with the prototype state equation minimally realized. In order to consider the effect of output on the modal mass evaluation, separate equations for both displacement and acceleration outputs are proposed. The effect of noise variation can be predicted using Eq. (4). When a TMD is installed in an arbitrary floor of an MDOF structure, how the modal mass is determined is an important factor in the design of TMD. The modal mass varies with the location of the TMD, and the variation can be predicted using Eq. (4). When a TMD is installed in an arbitrary floor of an MDOF structure, the equation of motion of the system, excluding the damping term for simplicity, can be constructed as (Fig. 2)

\[
\begin{bmatrix}
M & 0 \\
0 & m_t
\end{bmatrix} \ddot{x} + \begin{bmatrix}
K + S k_t S^T & -S k_t \\
-S k_t & k_t
\end{bmatrix} x = \begin{bmatrix}
E \\
0
\end{bmatrix} f
\]

(5)

where, \(M\) and \(K\) are the mass and stiffness matrix of the structure, respectively; \(m_t\) and \(m_k\) are the mass and stiffness of the TMD; \(S\) is the matrix representing the location of the TMD; and \(x\) and \(y\) are the vectors of relative displacement of the structure and the TMD.

2. Modal Mass Computation

2.1. Modal mass from eigenvector normalization

The modal mass of an MDOF system with \(n\) discrete masses can be estimated approximately as

\[
M_i = \sum_{j=1}^{n} m_j \phi_{i,j}^2
\]

(1)

where \(m_j\) and \(\phi_{i,j}\) are the mass of the \(i\)-th DOF and the \(i\)-th component of the mode shape vector, respectively, as shown in Fig. 1.

The modal mass obtained above depends on how the modal vectors are determined. If the \(i\)-th component of the mode shape vector is normalized into a unit value, the modal mass is obtained as

\[
M_i = m_i \left( \frac{\phi_i}{\phi_1} \right)^2 + m_{i-1} \left( \frac{\phi_i}{\phi_2} \right)^2 + \ldots + m_1 \left( \frac{\phi_i}{\phi_n} \right)^2
\]

(2)

When the \(j\)-th component of the mode shape vector is normalized into a unit value, the modal mass is obtained as

\[
M_j = m_i \left( \frac{\phi_j}{\phi_j} \right)^2 + m_{i-1} \left( \frac{\phi_j}{\phi_j} \right)^2 + \ldots + m_1 \left( \frac{\phi_j}{\phi_j} \right)^2
\]

(3)

If \(\phi_i^2\) and \(\phi_j^2\) are multiplied to Eqs. (2) and (3), respectively, the following relationship holds true.

\[
M_i = \left( \frac{\phi_i}{\phi_j} \right)^2 M_j
\]

(4)

Equation (4) demonstrates that the modal mass obtained by normalization of \(i\)-th component of the mode shape vector is proportional to that obtained by normalization of \(j\)-th component of the mode shape vector multiplied by the square of the relative magnitude of the \(j\)-th and \(i\)-th components of the mode shape vector.

2.2. Modal mass of a structure with a TMD

The mass of a TMD is generally determined as 1-2 % of the first modal mass of a main structure. As the mass of the damper depends on the modal mass of the structure, how the modal mass is determined is an important factor in the design of TMD. The modal mass varies with the location of the TMD, and the variation can be predicted using Eq. (4). When a TMD is installed in an arbitrary floor of an MDOF structure, the equation of motion of the system, excluding the damping term for simplicity, can be constructed as (Fig. 2)
respectively, \( E \) is the unit matrix having the dimension of the structure, and \( f \) is the external load acting on a nodal point of the structure. Since TMD is usually applied to reduce the response of a specific mode, the equation of motion of the system is simplified for the specific mode as follows assuming the effect of other modes are negligible.

\[
\begin{bmatrix}
\bar{M}_i & 0 \\
0 & m_j
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_i \\
\dot{\phi}_j
\end{bmatrix}
+ \begin{bmatrix}
\bar{M}_i \omega_1^2 + k_i \phi_1^2 - \phi_{n,i} k_i \\
- \phi_{n,i} k_i
\end{bmatrix}
\eta_i
= \begin{bmatrix}
I \\
0
\end{bmatrix}
\ddot{f}_i
\]

(6)

where \( \bar{M}_i, \omega_1, \ddot{f}_i, \eta_i \) are the mass, natural frequency, generalized force, and generalized displacement of the first mode.

If the modal mass of the system is defined as the mass when the component of the mode shape vector corresponding to the location of the TMD is normalized as one, i.e. =1, then it can be noticed in Eq. (6) that the dynamic property of the TMD becomes independent of the mode shape of the structure. This enables the estimation of modal mass of the structure in accordance with the location of TMD. If the generalized modal mass of a structure obtained by normalizing the largest component of the mode vector into one is \( \bar{M}_{HH} \), the modal mass of the system when a TMD is installed, \( \bar{M}_i \), can be obtained as follows using the mode shape component, \( \phi_i \), corresponding to the DOF where the TMD is installed.

\[
\bar{M}_i = \left( \frac{\phi_i}{\bar{M}_{HH}} \right)^2 \bar{M}_{HH} = \frac{1}{\phi_i^2} \bar{M}_{HH}
\]

(7)

As the vibration control effect of a TMD generally increases with increasing ratio of the mass of TMD to the structural modal mass, the smaller modal mass would be beneficial for vibration reduction if the mass of the device is constant. Therefore it can be seen from Eq. (7) that the effect of the location of a TMD is significant, and that the modal mass can be used as a parameter to represent the location of the TMD.

### 2.3 Modal mass from system identification method

The dynamic properties of a structure predicted by eigenvalue analysis may be different from the exact values due mainly to the assumption associated with the analytical modeling. The precise dynamic properties can be measured by experiments after the construction of the structure is completed. The mass of a structure can be estimated nearly precisely by simple computation, but the modal mass is a dynamic property related not only to mass distribution but also to mode shape vectors. In this study the system identification technique is extended to obtain the modal mass of a structure.

System identification is a technique to identify dynamic properties of a structure such as transfer functions. Various methods have been developed both in time and in frequency domains according to characteristics of inputs and outputs. Also various models such as ARX, ARMAX, OE, and Box-Jenkins are used in accordance with order and shape of polynomial to represent the transfer functions of the system. The identification model can be expressed in terms of either polynomial or state equation in discrete or continuous time domain and frequency domain.

The system identification model used in this study to estimate the modal mass is the state-space equation model in continuous time-domain. The state-space equation can be obtained directly or indirectly from system identification. The system identification model provides a transfer function that includes higher modes depending on its order. Therefore the identification result needs to be compressed into that of SDOF system in order to obtain the modal mass. In this study the identified state-space equation is compressed by \( H^\infty \) optimal model reduction method to realize only the first mode. As the state equation is affected by the factors such as input, output, and the noise associated with the output, the effect of those quantities on the modal mass is also considered in the formulation. Formulations of modal mass for both displacement and acceleration responses are derived.

The state-space equation derived using the time-histories of the input excitation and the recorded response output is presented in Eq. (8). In the state equation the first mode is assumed to be realized by the \( H^\infty \) optimal model reduction of the system identification model with high order transfer function.

\[
\begin{align}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align}
\]

(8a, b)

where \( x \) is the 2x1 state variable vector; \( u \) and \( y \) are the input and output of the structure, respectively; and \( A, B, C, D \) are the system matrices with dimensions of 2x2, 2x1, 1x2 and 1x1, respectively, obtained from system identification. In the state-space equation such as Eq. (8), there can be infinite number of solutions and system matrices that satisfy the relation between input \( u \) and output \( y \). Therefore the state variable has no physical meaning. However there exists a state equation in which the state variable retains physical meaning. Such a state equation can be obtained in the process of transforming the equation of motion of a SDOF system into state-space equation. The equation of motion of a SDOF system is as

\[
M \ddot{x} + C \dot{x} + Kx = f
\]

(9)
The state equation of Eq. (8) obtained from system identification can be transformed into prototype state equation of Eq. (10) using the matrix transformation. The state variable $\bar{z}$ in Eq. (8) can be transformed into $z$ in Eq. (10) using a transformation matrix $T$ such that

$$z = T\bar{z}$$

(14)

Substituting Eq. (14) into Eq. (10) results in

$$\dot{\bar{z}} = T^{-1}AT\bar{z} + T^{-1}Bu$$

(15,a)

$$y = CT\bar{z} + Du$$

(15,b)

In comparison with Eq. (8), the system matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ correspond to

$$\bar{A} = T^{-1}AT$$

(16)

$$\bar{B} = T^{-1}B$$

$$\bar{C} = CT^{-1}$$

$$\bar{D} = D$$

Equation (16) implies that a system identified in arbitrary format can be transformed into prototype state equation with variables having physical meaning using the transformation matrix $T$. In order to obtain the transformation matrix $T$, Eq. (16) is modified as

$$AT = \bar{T}\bar{A}$$

(17,a)

$$B = \bar{T}\bar{B}$$

(17,b)

Multiplying the matrix $A$ to the left-hand-side of Eq. (17,b) and using the relation of Eq. (17,a), the following relation holds.

$$AB = \bar{T}\bar{A}\bar{B}$$

(18)

Since Eqs. (17,b) and (18) are column vectors with dimension of 2 x 1, the controllability matrix can be obtained as follows rearranging those equation to obtain the transformation matrix $T$.

$$[B\ AB] = T[\bar{B}\ \bar{A}\bar{B}]$$

(19)

Similarly, the observability matrix can be obtained as

$$[C\ CA]T = [\bar{C}\ \bar{C}A]$$

(20)

From Eq. (19) or Eq. (20), the transformation matrix $T$ can be obtained as

$$T = [B\ AB][\bar{B}\ \bar{A}\bar{B}]^{-1}$$

(21,a)

$$T = [C\ CA]^{-1}[\bar{C}\ \bar{C}A]$$

(21,b)
Even though Eqs. (21,a) and (22,b) represent the same transformation matrix \( T \), the matrix \( B \) in Eq. (21,a) includes the unknown modal mass \( M \) as shown in Eq. (11), and therefore Eq. (21,a) cannot be used to compute \( T \). On the contrary, the matrices \( C, A, \bar{C}, \) and \( \bar{A} \) in (21,b) can be easily obtained from system identification, which leads to the transformation matrix \( T \).

Substituting \( T \) of Eq. (21,b) into Eq. (19) results in

\[
[B \ AB] = \left[ \frac{C}{CA} \right]^{-1} \left[ \frac{\bar{C}}{CA} \right] [B \ \bar{AB}] \tag{22}
\]

The right-hand-side of Eq. (22) is composed of matrices which can be obtained by conventional system identification. Whereas the matrices in the left-hand-side is available after being transformed into the state equation of a SDOF system. The matrices \( A \) and \( B \) are rewritten here for convenience.

\[
[B \ AB] = [B_o \ AB_o]^i \left[ \frac{C}{CA} \right]^{-1} \left[ \frac{\bar{C}}{CA} \right] [B \ \bar{AB}] \tag{23}
\]

where \( B_o = [0 \ 1]^T \). Substituting Eq. (23) into Eq. (22) and rearranging leads to

\[
\frac{1}{M} E = [B_o \ AB_o]^i \left[ \frac{C}{CA} \right]^{-1} \left[ \frac{\bar{C}}{CA} \right] [B \ \bar{AB}] \tag{24}
\]

where \( E \) is the 2x2 identity matrix. In Eq. (24) the matrices \( \bar{A}, \bar{B}, \) and \( \bar{C} \) can be obtained from system identification and the matrices \( A, B_o, \) and \( C \) can be expressed by the natural frequency and damping ratio estimated by system identification (i.e. \( \omega_1 \) and \( \xi_1 \)). Consequently, the modal mass \( \bar{M} \) is readily available.

3. Numerical Examples

3.1. Modal mass of a SDOF system

The proposed method for estimating the modal mass is applied to a SDOF system subjected to an external excitation \( F(t) \) shown in Fig. 3. A low-pass filtered white noise is used for an input excitation. The dynamic properties of the system and the characteristics of the input excitation are presented in Table 1. The time interval of the excitation is 0.01 second and the total duration is 600 seconds. The system matrices (\( \bar{A}, \bar{B}, \bar{C}, \) and \( D \)) and the dynamic characteristics identified using the displacement and acceleration outputs are summarized in Table 2 and 3, respectively.

![Fig. 3. SDOF system](image)

<table>
<thead>
<tr>
<th>Table 1. SDOF system and input excitation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SDOF System</strong></td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>System matrices</strong></td>
</tr>
<tr>
<td>( B = \frac{1}{M} \begin{bmatrix} 1 \ 1 \end{bmatrix} )</td>
</tr>
<tr>
<td>( C = [1 \ 0]^T ) (Disp.)</td>
</tr>
<tr>
<td>( \bar{C} =\begin{bmatrix} -19.3444 \ -0.044 \end{bmatrix} ) (Acc.)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Excitation</th>
<th>Type</th>
<th>Low-pass filtered white noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude</td>
<td>( \pm 100 ) when normalized to the structural mass</td>
<td></td>
</tr>
<tr>
<td>Filter bandwidth</td>
<td>Less than 2 Hz</td>
<td></td>
</tr>
<tr>
<td>Filter transfer function</td>
<td>( 0.0036s^2 + 0.0072s + 0.0036 )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{1.0000s^2 - 1.8227s + 0.8372} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Analysis condition | Time step | 0.01 sec. | Duration | 600 sec. |

<table>
<thead>
<tr>
<th>Table 2. Modal mass estimation results using displacement response</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>System matrices</strong></td>
</tr>
<tr>
<td>( A = \begin{bmatrix} 99.9135 &amp; 100.0543 \ -100.0102 &amp; -99.9577 \end{bmatrix} )</td>
</tr>
<tr>
<td>( B = 10^{-3} \begin{bmatrix} 0.1279 \ -0.0427 \end{bmatrix} )</td>
</tr>
<tr>
<td>( C = [1 \ 0]^T )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3. Modal mass estimation results using acceleration response</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>System matrices</strong></td>
</tr>
<tr>
<td>( A = \begin{bmatrix} 99.9135 &amp; 100.0543 \ -100.0102 &amp; -99.9577 \end{bmatrix} )</td>
</tr>
<tr>
<td>( B = 10^{-3} \begin{bmatrix} -0.2452 \ 0.0818 \end{bmatrix} )</td>
</tr>
<tr>
<td>( C = [1 \ 0]^T )</td>
</tr>
</tbody>
</table>

It can be observed that the natural frequency and the damping ratio obtained from the system matrices are almost identical to those obtained from eigenvalue
analysis. The modal mass computed using Eq. (24) for displacement and acceleration output matches the true mass of the system with an error of 0.26 % and 2.2 %, respectively.

Then the effect of noise included in the output on the evaluation of the modal mass is also investigated. The acceleration response is used for output, and the added noise is white noise with its RMS amplitude increasing up to 25 % of the acceleration output response.

Table 4. Effect of noise on the modal mass estimation

<table>
<thead>
<tr>
<th>Noise level</th>
<th>Modal mass</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>1184 ton</td>
<td>-</td>
</tr>
<tr>
<td>7%</td>
<td>1184.8 ton</td>
<td>-</td>
</tr>
<tr>
<td>15%</td>
<td>1185.8 ton</td>
<td>0.1%</td>
</tr>
<tr>
<td>17%</td>
<td>1186.1 ton</td>
<td>0.18%</td>
</tr>
<tr>
<td>18%</td>
<td>-30.2709</td>
<td>Unstable</td>
</tr>
<tr>
<td>20%</td>
<td>-31.7509</td>
<td>Unstable</td>
</tr>
</tbody>
</table>

Table 4 presents the modal mass obtained from output response including various levels of noise. It can be noticed that up to a certain level of noise (17 % in this case) the modal mass can be estimated quite precisely. However for higher level of noise the prediction of modal mass becomes unstable. It can be concluded that the magnitude of noise influences the stability of observability and controllability matrices, and large noise leads to unreasonable value for modal mass.

3.2. Modal mass of an MDOF system

The MDOF system used in the numerical analysis is the 5-story shear building shown. The fundamental modal mass obtained using the fundamental mode shape vector normalized in such a way that the top-floor component is one is 1170 ton. The other dynamic characteristics and input excitation are summarized in Table 5. The input excitation, which is the same white noise time history used in the previous example, is assumed to act on the top floor of the structure.

Table 5. MDOF system and input excitation

<table>
<thead>
<tr>
<th>Story mass</th>
<th>416.84 ton</th>
</tr>
</thead>
<tbody>
<tr>
<td>Story damping</td>
<td>227,270 N sec/m</td>
</tr>
<tr>
<td>Story stiffness</td>
<td>100 MN/m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Modal characteristics</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (ton)</td>
<td>1170</td>
<td>1385</td>
<td>2007</td>
<td>3992</td>
<td>14442</td>
</tr>
<tr>
<td>Damping (%)</td>
<td>0.5</td>
<td>1.46</td>
<td>2.31</td>
<td>2.96</td>
<td>3.38</td>
</tr>
<tr>
<td>Frequency (Hz)</td>
<td>0.701</td>
<td>2.048</td>
<td>3.228</td>
<td>4.147</td>
<td>4.731</td>
</tr>
<tr>
<td>Response output</td>
<td>Disp. and acc. of top floor</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Input excitation</td>
<td>Same as Ex. 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Numerical analysis has been carried out to compute the displacement and the acceleration at the top story. With the input excitation and the output responses, a state space equation model of order 10 is used for system identification. The results are transformed into the state equation of order 2 using the $H_{\infty}$ optimal model reduction method to obtain the first modal mass. Table 6 presents the condensed model and the modal mass obtained using the top-floor displacement and acceleration responses. As in the previous example, the error associated with the modal mass is larger when the acceleration is used in the system identification. This is due to the fact that the contribution of higher modes plays a role similar to noise when the acceleration is used.

Table 6. Condensed model and modal mass estimation results

<table>
<thead>
<tr>
<th>Response output</th>
<th>Condensed system matrices</th>
<th>Fundamental Modal mass (error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top floor disp.</td>
<td>$\mathbf{A} = \begin{bmatrix} -0.0223 &amp; 4.4085 \ -4.4085 &amp; -0.0219 \end{bmatrix}$</td>
<td>1178.1 ton (+0.69%)</td>
</tr>
<tr>
<td>Top floor acc.</td>
<td>$\mathbf{A} = \begin{bmatrix} -0.0405 &amp; 4.4085 \ -4.4085 &amp; -0.0037 \end{bmatrix}$</td>
<td>1302.2 ton (+11.3%)</td>
</tr>
</tbody>
</table>

3.3. Modal mass of structures with TMD

In this section the modal mass of a structure with a TMD is evaluated and the variation of the modal mass with varying TMD location is investigated. To vibrate the structure the TMD is excited first with forcing frequency equal to the tuned natural frequency. As the TMD is tuned to the natural frequency of the structure, even small movement of the TMD can easily excite the structure.

If it is assumed that the structure is excited only in the tuned mode and the contribution of the other modes are negligible, the equation of motion in Eq. (10) can be simplified as

$$\begin{bmatrix} \ddot{y}_1 + k_1 y_1 \\ \ddot{y}_2 + k_2 y_2 \\ \ddots \ddots \ddots \ddots \\ \ddot{y}_n + k_n y_n \end{bmatrix} - \begin{bmatrix} \ddot{u}_r \\ \ddot{u}_r \\ \ddots \ddots \ddots \ddots \\ \ddot{u}_r \end{bmatrix} - \begin{bmatrix} m_1 \dddot{y}_1 \\ m_2 \dddot{y}_2 \\ \ddots \ddots \ddots \ddots \\ m_n \dddot{y}_n \end{bmatrix} = \begin{bmatrix} -U_r \\ -U_r \\ \ddots \ddots \ddots \ddots \\ -U_r \end{bmatrix}$$

(25,a)

$$m_i \dddot{y}_i - k_i y_i - k_i y_i = -U_r$$

(25,b)
and the TMD is installed at the height of 72 m from
the ground. The dynamic characteristics of the
structure computed numerically in the design stage
and the properties of the TMD are presented in Table 8.

First, a five-story shear building with TMD is
analyzed in order to investigate the effect of TMD
location on the modal mass estimation. The mass of
the TMD is assumed to be 11.7 ton which is 1 % of the
first mode modal mass, which is 1170 ton. A harmonic
excitation force with its frequency equal to the tuned
natural frequencies acting between the TMD and the
structure is used as an input. Table 7 compares the
identified modal masses and those obtained from
eigenvalue analysis, and it can be observed that they
agree quite well. It also can be observed that the modal
mass increases as the location of the TMD moves
down, which results in decrease on mass ratio of TMD
to the generalized mass. This implies that the
effectiveness of TMD changes depending on the TMD
location, and that the modal mass can be used as a
parameter to represent the effectiveness of TMD
qualitatively.

Next, the procedure for estimating modal mass is
applied to a real structure with TMD. The structure,
shown in Fig. 4, is a control tower of the Yang-Yang
International Airport. The height of the tower is 80.1 m

<table>
<thead>
<tr>
<th>TMD location (floor)</th>
<th>System matrices</th>
<th>Modal mass (ton)</th>
<th>Estimated modal mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>$A = \begin{bmatrix} 14.6957 &amp; -0.3510 \ 668.7754 &amp; 14.6515 \end{bmatrix}$</td>
<td>14,442</td>
<td>14,450 (0.06%)</td>
</tr>
<tr>
<td></td>
<td>$B = \begin{bmatrix} 0.0057 \ 0.0348 \end{bmatrix}$</td>
<td>14,442</td>
<td>14,450 (0.06%)</td>
</tr>
<tr>
<td></td>
<td>$C = \begin{bmatrix} 0.0738 \ -0.0019 \end{bmatrix}$</td>
<td>14,442</td>
<td>14,450 (0.06%)</td>
</tr>
<tr>
<td>3rd</td>
<td>$A = \begin{bmatrix} -1.8359 &amp; -2.8344 \ 8.0179 &amp; 1.7918 \end{bmatrix}$</td>
<td>2,007</td>
<td>2,013 (0.3%)</td>
</tr>
<tr>
<td></td>
<td>$B = \begin{bmatrix} 0.0057 \ 0.0348 \end{bmatrix}$</td>
<td>2,007</td>
<td>2,013 (0.3%)</td>
</tr>
<tr>
<td></td>
<td>$C = \begin{bmatrix} 0.0752 \ -0.0134 \end{bmatrix}$</td>
<td>2,007</td>
<td>2,013 (0.3%)</td>
</tr>
<tr>
<td>5th</td>
<td>$A = \begin{bmatrix} -3.4714 &amp; 3.7772 \ -8.2943 &amp; 3.4273 \end{bmatrix}$</td>
<td>1,170</td>
<td>1,171.3 (0.11%)</td>
</tr>
<tr>
<td></td>
<td>$B = \begin{bmatrix} -0.0057 \ -0.0424 \end{bmatrix}$</td>
<td>1,170</td>
<td>1,171.3 (0.11%)</td>
</tr>
<tr>
<td></td>
<td>$C = \begin{bmatrix} 0.1277 \ -0.0137 \end{bmatrix}$</td>
<td>1,170</td>
<td>1,171.3 (0.11%)</td>
</tr>
</tbody>
</table>

Fig. 4. Yang-Yang International airport Control tower

(a) Acceleration of tower

(b) Acceleration of TMD

Fig. 5. Acceleration time history
Right after the construction of the structure was finished, experiments were carried out to evaluate the effectiveness of the TMD. First the TMD was excited manually, and then the inertia force of the moving mass excited the tower. During the experiment, the acceleration of the tower and the TMD were measured, and the relative displacement between the tower and the TMD was also recorded. Fig. 5 presents the acceleration time history of the tower and the TMD used in the system identification. Table 9 summarizes the results of the system identification and the estimated modal mass. In comparison with Table 8, it can be observed that the dynamic characteristics of the structure obtained from system identification are somewhat different from those obtained from eigenvalue analysis. In case of modal mass, the difference is about 8 %.

### Table 9. System identification of control tower

<table>
<thead>
<tr>
<th>System matrices</th>
<th>Dynamic Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A} = \begin{bmatrix} -0.0271 &amp; -2.2928 \ 2.2928 &amp; -0.0247 \end{bmatrix}$</td>
<td>Mass (ton) 1756 (+9%)</td>
</tr>
<tr>
<td>$\mathcal{B} = 10^{-3} \begin{bmatrix} 0.8310 \ -0.8310 \end{bmatrix}$</td>
<td>Freq. (Hz) 0.3645 (+5.6%)</td>
</tr>
<tr>
<td>$\mathcal{C} = 10^{-3} \begin{bmatrix} -0.8310 &amp; -0.7847 \end{bmatrix}$</td>
<td>Damping ratio (%) 1.13</td>
</tr>
</tbody>
</table>

### 4. Conclusions

In this study a new method is proposed to estimate the modal mass of a structure by system identification. The high order state equation constructed by system identification is reduced to state equation of order two through $H^\infty$ optimal model reduction technique. The modal mass was obtained in comparison with the prototype state equation of a SDOF system. To validate the applicability of the proposed method, numerical analyses were carried out with a SDOF and 5-story MDOF structures. The method was further verified by analysis of a structure with TMD.

The numerical analysis results show that the proposed method estimates the modal mass of a structure quite precisely. The effect of noise in the output was not significant if the noise is not large, while the estimated modal mass became unstable when the noise is larger than a certain level. The numerical examples of MDOF structures with TMD show that the modal mass estimated by system identification depends on the TMD location, which implies that the modal mass can be used as a parameter to determine the optimal location of the TMD and the optimal design parameter of the TMD. Finally the analysis and experiment of a control tower with a TMD showed that the modal mass of a structure with a TMD could be estimated with reasonable accuracy using the acceleration responses of the structure and the TMD obtained from vibration test.

### 5. References


### Acknowledgment

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